

On irreducible algebraic sets over linearly ordered semilattices II

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Abstract

Equations over linearly ordered semilattices are studied. For any equation $t(X) = s(X)$ we find irreducible components of its solution set and compute the average number of irreducible components of all equations in n variables.

1 Introduction

This paper is devoted to the following problem. One can define a notion of an equation over a linearly ordered semilattice $L_l = \{a_1, a_2, \dots, a_l\}$ (the formal definition of an equation is given below in the paper). A set Y is *algebraic* if it is the solution set of some system of equations over L_l . Let us consider an equation $t(X) = s(X)$ in n variables over L_l , and Y be the solution set of $t(X) = s(X)$. One can find algebraic sets Y_1, Y_2, \dots, Y_m such that $Y = \bigcup_{i=1}^m Y_i$. One can decompose each Y_i into a union of other algebraic sets, etc. This process terminates after a finite number of steps and gives a decomposition of Y into a union of *irreducible* algebraic sets Y_i (the sets Y_i are called the *irreducible components* of Y). Roughly speaking, irreducible algebraic sets are “atoms” which form any algebraic set. The size and the number of such “atoms” are important characteristics of the semilattices L_l , since there are connections between irreducible algebraic sets and universal theory of linearly ordered semilattices (see [1]). Moreover, the number of irreducible components was involved in the estimation of lower bounds of algorithm complexity (see [2] for more details).

In this paper we assume $n \leq l$ (i.e. the order of the semilattice L_l is not less than the number of variables in $t(X) = s(X)$) and study (Section 4) the properties of algebraic sets over L_l . Precisely, for any equation $t(X) = s(X)$ in n variables we count the number of irreducible components (see (8)), and in Section 5 we count the average number $\overline{\text{Irr}}(n)$ of irreducible components of the solution sets of equations in n variables.

Remark that the current paper is the sequel of [3], where we solved the similar problems assuming $n > l$ (we discuss this case in Remark 2.1 below).

2 Main definitions

Let $L_l = \{a_1, a_2, \dots, a_l\}$ be the linearly ordered semilattice of l elements and $a_1 < a_2 < \dots < a_l$. The multiplication in L_l is defined by $a_i \cdot a_j = a_{\min(i,j)}$. Obviously, the linear order on L_l can be expressed by the multiplication as follows

$$a_i \leq a_j \Leftrightarrow a_i a_j = a_i.$$

A *term* $t(X)$ in variables $X = \{x_1, x_2, \dots, x_n\}$ is a commutative word in letters x_i .

Let $\text{Var}(t)$ be the set of all variables occurring in a term $t(X)$. Following [1], an *equation* is an equality of terms $t(X) = s(X)$. Below we consider inequalities $t(X) \leq s(X)$ as equations, since $t(X) \leq s(X)$ is the short form of $t(X)s(X) = t(X)$. Notice that we consider equations as *ordered pairs* of terms, i.e. the expressions $t(X) = s(X)$, $s(X) = t(X)$ are *different* equations. Let $Eq(n)$ denote the set of all equations in $X = \{x_1, x_2, \dots, x_n\}$ variables (we assume that each $t(X) = s(X) \in Eq(n)$ contains the occurrences of all variables x_1, x_2, \dots, x_n). An equation $t(X) = s(X) \in Eq(n)$ is said to be a (k_1, k_2) -*equation* if $|\text{Var}(t) \setminus \text{Var}(s)| = k_1$ and $|\text{Var}(s) \setminus \text{Var}(t)| = k_2$. For example, $x_1x_2 = x_1x_3x_4$ is a $(1, 2)$ -equation. Let $Eq(k_1, k_2, n) \subseteq Eq(n)$ be the set of all (k_1, k_2) -equations in n variables. Obviously,

$$Eq(n) = \bigcup_{(k_1, k_2) \in K_n} Eq(k_1, k_2, n), \quad (1)$$

where

$$K_n = \{(k_1, k_2) \mid k_1 + k_2 \leq n\} \setminus \{(0, n), (n, 0)\}.$$

Each equation $t(X) = s(X) \in Eq(k_1, k_2, n)$ is uniquely defined by k_1 variables in the left part and by k_2 other variables in the right part (the residuary $n - k_1 - k_2$ variables should occur in both parts of the equation). Thus,

$$\#Eq(k_1, k_2, n) = \binom{n}{k_1} \binom{n - k_1}{k_2}.$$

By (1), one can compute that

$$\#Eq(n) = 3^n - 2.$$

Remark 2.1. In this paper we consider only equations $t(X) = s(X)$ with $n \leq l$, i.e. the number of variables occurring in $t(X) = s(X)$ is not more than the order of the semilattice L_l . The case $n > l$ needs a completely different technic and was considered in [3]. All main results of the current paper do not hold for the case $n > l$.

A point $P \in L_l^n$ is a *solution* of an equation $t(X) = s(X)$ if $t(P), s(P)$ define the same element in the semilattice L_l . By the properties of linearly ordered semilattices, a point $P = (p_1, p_2, \dots, p_n)$ is a solution of $t(X) = s(X)$ iff there exist variables $x_i \in \text{Var}(t)$, $x_j \in \text{Var}(s)$ such that $p_i = p_j$ and $p_i \leq p_k$ for all $1 \leq k \leq n$. The set of all solutions of an equation $t(X) = s(X)$ is denoted by $V(t(X) = s(X))$.

An arbitrary set of equations is called a *system*. The set of all solutions $V(\mathbf{S})$ of a system $\mathbf{S} = \{t_i(X) = s_i(X) \mid i \in I\}$ is defined as $\bigcap_{i \in I} V(t_i(X) = s_i(X))$. A set $Y \subseteq L_l^n$ is called *algebraic over L_l* if there exists a system \mathbf{S} in n variables with $V(\mathbf{S}) = Y$. An algebraic set Y is *irreducible* if Y is not a proper finite union of other algebraic sets.

Proposition 2.2. ([3], Proposition 2.2) *Any algebraic set Y over L_l is a finite union of irreducible sets*

$$Y = Y_1 \cup Y_2 \cup \dots \cup Y_m, \quad Y_i \not\subseteq Y_j \text{ for all } i \neq j, \quad (2)$$

and this decomposition is unique up to a permutation of components.

The subsets Y_i from the union (2) are called the *irreducible components* of Y .

Let Y be an algebraic set over L_l defined by a system $\mathbf{S}(X)$. One can define an equivalence relation \sim_Y over the set of all terms in variables X as follows

$$t(X) \sim_Y s(X) \Leftrightarrow t(P) = s(P) \text{ for any point } P \in Y.$$

The set of all \sim_Y -equivalence classes is called *the coordinate semilattice of Y* and denoted by $\Gamma(Y)$ (see [1] for more details). The following statement describes the coordinate semilattices of irreducible algebraic sets.

Proposition 2.3. ([3], Proposition 2.3) *A set Y is irreducible over L_l iff $\Gamma(Y)$ is embedded into L_l*

There are different algebraic sets over L_l with isomorphic coordinate semilattices. Such sets are called *isomorphic*. For example, the following sets

$$Y_1 = V(\{x_1 \leq x_2 \leq x_3\}), \quad Y_2 = V(\{x_3 \leq x_2 \leq x_1\})$$

has the isomorphic coordinate semilattices

$$\Gamma(Y_1) = \langle x_1, x_2, x_3 \mid x_1 \leq x_2 \leq x_3 \rangle \cong L_3,$$

$$\Gamma(Y_2) = \langle x_1, x_2, x_3 \mid x_3 \leq x_2 \leq x_1 \rangle \cong L_3.$$

Thus, Y_1, Y_2 are isomorphic.

3 Example

Let $n = 3, l = 3$. We have exactly $Eq(3) = 3^3 - 2 = 25$ equations in three variables over L_3 . The following table contains the information about such equations over L_3 . The second column contains systems which define irreducible components of the solution set of an equation in the first column. A cell of the table contains \uparrow if an information in this cell is similar to the cell above.

Table 1.

| Equations | Irreducible components (IC) | Number of IC |
|---|--|--------------|
| $x_1x_2x_3 = x_1x_2x_3$ | $x_1 \leq x_2 \leq x_3 \cup x_1 \leq x_3 \leq x_2 \cup$ $x_2 \leq x_1 \leq x_3 \cup x_2 \leq x_3 \leq x_2 \cup$ $x_3 \leq x_1 \leq x_2 \cup x_3 \leq x_2 \leq x_1$ | 6 |
| $x_1 = x_1x_2x_3,$ $x_1x_2x_3 = x_1$ | $x_1 \leq x_2 \leq x_3 \cup x_1 \leq x_3 \leq x_1$ | 2 |
| $x_2 = x_1x_2x_3,$ $x_1x_2x_3 = x_2$ | \uparrow | 2 |
| $x_3 = x_1x_2x_3,$ $x_1x_2x_3 = x_3$ | \uparrow | 2 |
| $x_1 = x_2x_3,$ $x_2x_3 = x_1$ | $x_1 = x_2 \leq x_3 \cup x_1 = x_3 \leq x_2$ | 2 |
| $x_2 = x_1x_3,$ $x_1x_3 = x_2$ | \uparrow | 2 |
| $x_3 = x_1x_2,$ $x_1x_2 = x_3$ | \uparrow | 2 |
| $x_1x_2 = x_1x_3,$ $x_1x_3 = x_1x_2$ | $x_1 \leq x_2 \leq x_3 \cup x_1 \leq x_3 \leq x_2 \cup$ $x_2 = x_3 \leq x_1$ | 3 |
| $x_1x_2 = x_2x_3,$ $x_2x_3 = x_1x_2$ | \uparrow | 3 |
| $x_1x_3 = x_2x_3,$ $x_2x_3 = x_1x_3$ | \uparrow | 3 |
| $x_1x_2 = x_1x_2x_3,$ $x_1x_2x_3 = x_1x_2$ | $x_1 \leq x_2 \leq x_3 \cup x_1 \leq x_3 \leq x_2 \cup$ $x_2 \leq x_1 \leq x_3 \cup x_2 \leq x_3 \leq x_1$ | 4 |
| $x_1x_3 = x_1x_2x_3,$ $x_1x_2x_3 = x_1x_3$ | \uparrow | 4 |
| $x_2x_3 = x_1x_2x_3,$ $x_1x_2x_3 = x_2x_3$ | \uparrow | 4 |

Notice that $V(x_1 = x_2 \leq x_3)$ does not define an irreducible component for $Y = V(x_1x_2 = x_1x_3)$, since $V(x_1 = x_2 \leq x_3)$ is included into the solution set of another irreducible component $V(x_1 \leq x_2 \leq x_3)$. Similarly, $V(x_3 = x_1 \leq x_2)$ is not an irreducible component for Y , since it is contained in the irreducible component $V(x_1 \leq x_3 \leq x_2)$.

It turns out that the number of irreducible components does not depend on the semilattice order l . One can directly compute the average number of irreducible components of algebraic sets defined by equations in three variables:

$$\overline{\text{Irr}}(3) = \frac{6 + 2(2 + 2 + 2 + 2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 4)}{25} = \frac{72}{25} = 2.88 \quad (3)$$

Recall that in Section 5 we obtain the general expression for $\overline{\text{Irr}}(n)$ (10). Clearly, (10) will give (3) for $n = 3$.

4 Decompositions of algebraic sets

Let Y denote the solution set of an equation $t(X) = s(X)$ over the semilattice $L_l = \{a_1, a_2, \dots, a_l\}$. The table above shows that any irreducible component sorts the variables X into some order. The following definition formalizes this property of irreducible components.

Let σ be a permutation of the set $\{1, 2, \dots, n\}$; σ sorts the set X as follows $\{x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}\}$, i.e. $\sigma(i)$ is the i -th variable in the sorted set X . A permutation σ is called a *permutation of the first (second) kind* if $x_{\sigma(1)} \in \text{Var}(t) \cap \text{Var}(s)$ (respectively, $x_{\sigma(2)} \in \text{Var}(t) \setminus \text{Var}(s)$, $x_{\sigma(1)} \in \text{Var}(s) \setminus \text{Var}(t)$). Let $\chi(\sigma) \in \{1, 2\}$ denote the kind of a permutation σ .

Example 4.1. Let us consider an algebraic set $Y_0 = V(x_1x_2 = x_1x_3)$. By the table above, Y_0 is the union of the following irreducible components

$$Y_1 = V(x_1 \leq x_2 \leq x_3), Y_2 = V(x_1 \leq x_3 \leq x_2), Y_3 = V(x_2 = x_3 \leq x_1)$$

The irreducible components Y_1, Y_2, Y_3 define the following permutations

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

Moreover, σ_1, σ_2 are permutations of the first kind, whereas σ_3 is of the second kind.

A permutation σ defines an algebraic set Y_σ as follows:

$$Y_\sigma = V\left(\bigcup_{i=1}^{n-1} \{x_{\sigma(i)} \leq x_{\sigma(i+1)}\}\right) \quad (4)$$

if $\chi(\sigma) = 1$, and

$$Y_\sigma = V(\{x_{\sigma(1)} = x_{\sigma(2)}\} \bigcup_{i=2}^{n-1} \{x_{\sigma(i)} \leq x_{\sigma(i+1)}\}) \quad (5)$$

if $\chi(\sigma) = 2$.

Example 4.2. Let $\sigma_1, \sigma_2, \sigma_3$ be permutations from Example 4.1. Obviously, the sets $Y_{\sigma_1}, Y_{\sigma_2}, Y_{\sigma_3}$ defined by (4,5) coincide with the sets Y_1, Y_2, Y_3 respectively.

Lemma 4.3. Let $\chi(\sigma) \in \{1, 2\}$, then the set Y_σ is irreducible and moreover

$$\Gamma(Y_\sigma) \cong \begin{cases} L_n, & \text{if } \chi(\sigma) = 1 \\ L_{n-1}, & \text{if } \chi(\sigma) = 2 \end{cases} \quad (6)$$

Proof. By the definition of a coordinate semilattice, $\Gamma(Y_\sigma)$ is generated by the elements $\{x_1, x_2, \dots, x_n\}$ and has the following defined relations

$$x_{\sigma(1)} \leq x_{\sigma(2)} \leq \dots x_{\sigma(n)} \quad \text{if } \chi(Y_\sigma) = 1$$

and

$$x_{\sigma(1)} = x_{\sigma(2)} \leq \dots x_{\sigma(n)} \quad \text{if } \chi(Y_\sigma) = 2.$$

Thus, $\Gamma(Y_\sigma)$ is a linearly ordered semilattice, and (6) holds. By Proposition 2.3, the set Y_σ is irreducible. \square

The following lemma gives the irreducible decomposition of an algebraic set $Y = V(t(X) = s(X))$.

Lemma 4.4. *An algebraic set $Y = V(t(X) = s(X))$ is a union*

$$Y = \bigcup_{\chi(\sigma) \in \{1,2\}} Y_\sigma. \quad (7)$$

Proof. Suppose $P = (p_1, p_2, \dots, p_n) \in Y$. Let us sort p_i in the ascending order

$$p_{\sigma(1)} \leq p_{\sigma(2)} \leq \dots \leq p_{\sigma(n)},$$

where σ is a permutation of the set $\{1, 2, \dots, n\}$. We have that σ induces the sorting of the variable set X . Obviously, we may assume that $x_{\sigma(1)} \in \text{Var}(t)$ (if $x_{\sigma(1)} \notin \text{Var}(t)$, the properties of L_l provides an existence of a variable $x_{\sigma(i)} \in \text{Var}(t)$ such that $p_{\sigma(i)} = p_{\sigma(1)}$; in this case one can swap the values $\sigma(1)$ and $\sigma(i)$).

For example, the point $P = (a_2, a_1, a_1) \in V(x_1x_2 = x_1x_3)$ defines $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$ (the permutation obtained equals σ_3 from Example 4.1, so the point (a_2, a_1, a_1) belongs to the set Y_3).

Since σ is defined by the inequalities between the coordinates p_i , it follows $P \in Y_\sigma$.

Let us prove now $Y_\sigma \subseteq Y$ for each σ . Suppose $P = (p_1, p_2, \dots, p_n) \in Y_\sigma$. If $\chi(Y_\sigma) = 1$ then

$$x_{\sigma(1)} \in \text{Var}(t) \cap \text{Var}(s) \Rightarrow t(P) = s(P) = p_{\sigma(1)} \Rightarrow P \in V(t(X) = s(X)).$$

Otherwise ($\chi(Y_\sigma) = 2$), $t(P) = p_{\sigma(1)}$, $s(P) = p_{\sigma(2)}$, and (5) gives $p_{\sigma(1)} = p_{\sigma(2)}$. Therefore $P \in V(t(X) = s(X))$. \square

Lemma 4.5. *For distinct permutations σ, σ' we have $Y_\sigma \not\subseteq Y_{\sigma'}$ in (7).*

Proof. Let σ be a permutation of the first or second kind, and P_σ denote the following point

$$p_{\sigma(i)} = a_i \text{ if } \chi(\sigma) = 1,$$

and

$$p_{\sigma(i)} = \begin{cases} a_i, & 2 \leq i \leq n \\ a_2, & i = 1 \end{cases} \quad \text{if } \chi(\sigma) = 2.$$

For example, the permutations $\sigma_1, \sigma_2, \sigma_3$ from Example 4.1 define the points

$$P_1 = (a_1, a_2, a_3), \quad P_2 = (a_1, a_3, a_2), \quad P_3 = (a_3, a_2, a_2),$$

respectively.

Since P_σ preserves the order of variables, we have $P_\sigma \in Y_\sigma$.

Let us show now $P_\sigma \notin Y_{\sigma'}$ for every $\sigma' \neq \sigma$ (for example, each of the points P_1, P_2, P_3 above belong to a unique irreducible component from Example 4.1:

$$P_1 \in Y_1 \setminus (Y_2 \cup Y_3), \quad P_2 \in Y_2 \setminus (Y_1 \cup Y_3), \quad P_3 \in Y_3 \setminus (Y_1 \cup Y_2)).$$

There exists indexes $i < j$ such that $i = \sigma(\alpha)$, $j = \sigma(\beta)$, $i = \sigma'(\alpha')$, $j = \sigma'(\beta')$, with $\alpha < \beta$, $\alpha' > \beta'$. Hence the inequality $x_i \leq x_j$ holds in Y_σ , and $x_j \leq x_i$ holds in $Y_{\sigma'}$. Let us consider the following two cases:

1. If $\chi(\sigma) = 1$, then $p_i < p_j$ in P_σ , and we immediately obtain $P_\sigma \notin Y_{\sigma'}$.
2. Suppose $\chi(\sigma) = 2$. One should assume that $p_i = p_j = a_2$ (if $p_i < p_j$ we immediately obtain $P_\sigma \notin Y_{\sigma'}$). Then $\alpha = 1$, $\beta = 2$ and $i = \sigma(1)$, $j = \sigma(2)$ (one can similarly consider the case $i = \sigma(2)$, $j = \sigma(1)$). Hence $x_i \in \text{Var}(t) \setminus \text{Var}(s)$, $x_j \in \text{Var}(s) \setminus \text{Var}(t)$. By the definition of a permutation of the second kind, $\sigma'(1) = k \neq j$, and the inequality $x_k \leq x_j$ holds in $Y_{\sigma'}$. Let γ be the index such that $\sigma(\gamma) = k$. Since $\alpha = 1$, $\beta = 2$, we have $\gamma > 2$. Then $p_k = a_\gamma$, and $p_j < p_k$ for P_σ . Thus, $P \notin Y_{\sigma'}$.

□

According to Lemmas 4.3, 4.4, 4.5, we obtain the following statement.

Theorem 4.6. *The union (γ) is the irreducible decomposition of the set $Y = V(t(X) = s(X))$. The number of irreducible components is equal to the number of permutations of the first and second kind.*

5 Average number of irreducible components

One can directly compute that any (k_1, k_2) -equation admits

$$(n - k_1 - k_2)(n - 1)!$$

permutations of the first kind and

$$k_1 k_2 (n - 2)!$$

permutations of the second kind.

By Theorem 4.6, for a (k_1, k_2) -equation $t(X) = s(X)$ the number of its irreducible components equals

$$\text{Irr}(k_1, k_2, n) = (n - k_1 - k_2)(n - 1)! + k_1 k_2 (n - 2)! \quad (8)$$

The average number of irreducible components of algebraic sets defined by equations from $Eq(n)$ is

$$\begin{aligned} \overline{\text{Irr}}(n) &= \frac{\sum_{(k_1, k_2) \in K_n} \#Eq(k_1, k_2, n) \text{Irr}(k_1, k_2, n)}{\#Eq(n)} = \\ &= \frac{\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \#Eq(k_1, k_2, n) \text{Irr}(k_1, k_2, n) - \#Eq(0, n, n) \text{Irr}(0, n, n)}{\#Eq(n)}. \end{aligned}$$

Since

$$\text{Irr}(0, n, n) = (n - 0 - n)(n - 1)! + 0n(n - 2)! = 0,$$

we obtain

$$\overline{\text{Irr}}(n) = \frac{\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \#Eq(k_1, k_2, n) \text{Irr}(k_1, k_2, n)}{\#Eq(n)}.$$

Below we compute $\overline{\text{Irr}}$ using the following denotations:

1. $A \stackrel{(1)}{=} B$: an expression B is obtained from A by the binomial identity

$$a \binom{n}{a} = n \binom{n-1}{a-1}$$

2. $A \stackrel{(2)}{=} B$: an expression B is obtained from A by the following identity of binomial coefficients

$$\sum_{t=0}^n \binom{n}{t} t 2^t = 2n 3^{n-1}. \quad (9)$$

Let us demonstrate the proof of (9):

$$\sum_{t=0}^n \binom{n}{t} t 2^t \stackrel{(1)}{=} n \sum_{t=0}^n \binom{n-1}{t-1} 2^t = 2n \sum_{t=0}^n \binom{n-1}{t-1} 2^{t-1} = 2n \sum_{u=0}^{n-1} \binom{n-1}{u} 2^u = 2n 3^{n-1}$$

Let us compute $\overline{\text{Irr}}(n)$. We have that

$$\begin{aligned} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \#Eq(k_1, k_2, n) \text{Irr}(k_1, k_2, n) = \\ \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} (n-k_1-k_2)(n-1)! + k_1 k_2 (n-2)! = \\ n! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} - (n-1)! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} k_1 - \\ (n-1)! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} k_2 + (n-2)! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} k_1 k_2 = \\ S_1 - S_2 - S_3 + S_4, \end{aligned}$$

where

$$S_1 = n! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} = n! \sum_{k_1=0}^{n-1} \binom{n}{k_1} 2^{n-k_1} = n! (3^n - 1),$$

$$\begin{aligned} S_2 = (n-1)! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} k_1 = (n-1)! \sum_{k_1=0}^{n-1} \binom{n}{k_1} k_1 2^{n-k_1} \stackrel{(1)}{=} \\ n! \sum_{k_1=0}^{n-1} \binom{n-1}{k_1-1} 2^{n-k_1} = n! \sum_{t=0}^{n-2} \binom{n-1}{t} 2^{n-1-t} = \\ n! \left(\sum_{t=0}^{n-1} \binom{n-1}{t} 2^{n-1-t} - 1 \right) = n! (3^{n-1} - 1), \end{aligned}$$

$$\begin{aligned} S_3 = (n-1)! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} k_2 \stackrel{(1)}{=} \\ (n-1)! \sum_{k_1=0}^{n-1} \binom{n}{k_1} (n-k_1) \sum_{k_2=0}^{n-k_1} \binom{n-k_1-1}{k_2-1} = (n-1)! \sum_{k_1=0}^{n-1} \binom{n}{k_1} (n-k_1) 2^{n-k_1-1} = \\ (n-1)! \sum_{t=0}^n \binom{n}{t} t 2^{t-1} = \frac{(n-1)!}{2} \sum_{t=0}^n \binom{n}{t} t 2^t \stackrel{(2)}{=} n! 3^{n-1}, \end{aligned}$$

$$\begin{aligned}
S_4 &= (n-2)! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} k_1 k_2 \stackrel{(1)}{=} \\
&= (n-2)! \sum_{k_1=0}^{n-1} \binom{n}{k_1} k_1 (n-k_1) \sum_{k_2=0}^{n-k_1} \binom{n-k_1-1}{k_2-1} = (n-2)! \sum_{k_1=0}^{n-1} \binom{n}{k_1} k_1 (n-k_1) 2^{n-k_1-1} = \\
&= \frac{(n-2)!}{2} \sum_{k_1=0}^n \binom{n}{k_1} k_1 (n-k_1) 2^{n-k_1} = \frac{(n-2)!}{2} \sum_{t=0}^n \binom{n}{t} t(n-t) 2^t = \\
&= \frac{(n-2)!}{2} \left(n \sum_{t=0}^n \binom{n}{k_1} t 2^t - \sum_{t=0}^n \binom{n}{t} t^2 2^t \right) \stackrel{(2)}{=} \frac{(n-2)!}{2} (2n^2 3^{n-1} - S_5),
\end{aligned}$$

and

$$\begin{aligned}
S_5 &= \sum_{t=0}^n \binom{n}{k_1} t^2 2^t \stackrel{(1)}{=} n \sum_{t=0}^n \binom{n-1}{t-1} t 2^t = n \left(\sum_{t=0}^n \binom{n-1}{t-1} (t-1) 2^t + \sum_{t=0}^n \binom{n-1}{t-1} 2^t \right) = \\
&= n \left(2 \sum_{t=0}^n \binom{n-1}{t-1} (t-1) 2^{t-1} + \sum_{t=0}^n \binom{n-1}{t-1} 2^t \right) \stackrel{(2)}{=} n (4(n-1) 3^{n-2} + 2 \cdot 3^{n-1})
\end{aligned}$$

Finally, we obtain that

$$\begin{aligned}
S_1 - S_2 - S_3 + S_4 &= n!(3^n - 1) - n!(3^{n-1} - 1) - n!3^{n-1} + \\
&= \frac{(n-2)!}{2} (2n^2 3^{n-1} - n(4(n-1) 3^{n-2} + 2 \cdot 3^{n-1})) = n!3^{n-1} + (n-2)! 3^{n-2} n (3n - 2(n-1) - 3) = \\
&= n!3^{n-1} + n!3^{n-2} = 4n!3^{n-2}
\end{aligned}$$

and

$$\overline{\text{Irr}}(n) = \frac{4n!3^{n-2}}{3^n - 2} \sim \frac{4}{9}n! \quad (10)$$

Notice that the final answer does not depend on l if $l \leq n$. In particular, (10) gives

$$\overline{\text{Irr}}(3) = \frac{72}{25} = 2.88 \quad (11)$$

for $n = 3$, and (11) obviously coincides with (3).

References

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